CARINGBAH HIGH SCHOOL

2014

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

SECTION 1 (10 marks)
Attempt Questions 1 - 10
Allow about 15 minutes for section.
Use the multiple choice answer sheet for questions 1 - 10

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1. If w is a non-real cube root of unity the value $\frac{1}{1+w}+\frac{1}{1+w^2}$ is equal to

(A) -1 (B) 0 (C) 1 (D) none of these

2. What is the remainder when x^3+x^2+5x+6 is divided by $x+i$

(A) $7-4i$ (B) $7-6i$ (C) $5-4i$ (D) $5+6i$

3. The gradient of the tangent to $xy^3+2y=4$ at the point $(2,1)$ is

(A) -8 (B) $\frac{1}{8}$ (C) 8 (D) $\frac{-1}{8}$

4. The eccentricity of the ellipse $3x^2+5y^2-15=0$ is

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{2}{5}}$ (C) $\sqrt{\frac{8}{5}}$ (D) $\sqrt{\frac{5}{8}}$

5. The polynomial $3x^3-2x^2+x-7=0$ has roots α,β,γ.
Which polynomial has roots $\frac{2}{\alpha},\frac{2}{\beta},\frac{2}{\gamma}$?

(A) $3x^3-4x^2+4x-56$ (B) $7x^3-2x^2+8x-24=0$

(C) $9x^3-2x^2-27x-49=0$ (D) $24x^3-8x^2+2x-7=0$

6. The arg of iz where $z=1+i$ is

(A) $\frac{-\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $\frac{-3\pi}{4}$

7. Find $\int x\sin(x^2+3)dx$

(A) $-\frac{1}{2}\cos(x^2+3)+C$

(B) $-\frac{1}{2}\sin(x^2+3)+C$

(C) $\frac{1}{2}\cos(x^2+3)+C$

(D) $2x\cos(x^2+3)+C$

8. The polynomial equation $P(x)=0$ has real coefficients, and has roots which include

$$x=-2+i \quad \text{and} \quad x=2.$$

What is the minimum possible degree of $P(x)$?

(A) 1

(B) 2

(C) 3

(D) 4

9. What is the value of $\int_0^1 \frac{e^x}{1+e^x}dx$

(A) $\log_e(1+e)$

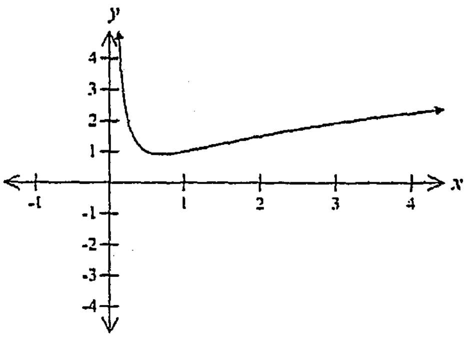
(B) 1

(C) $\log_e\left(\frac{1+e}{2}\right)$

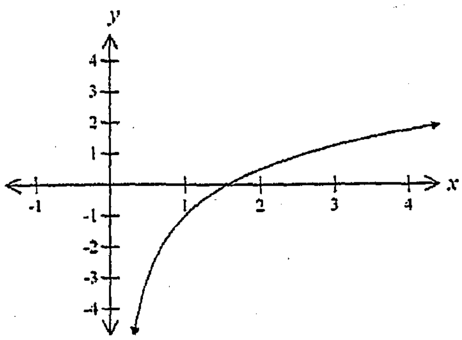
(D) $\log_e\frac{e}{2}-2$

10. Which of the following is the sketch of $y=\log_2 x+\frac{1}{x}$?

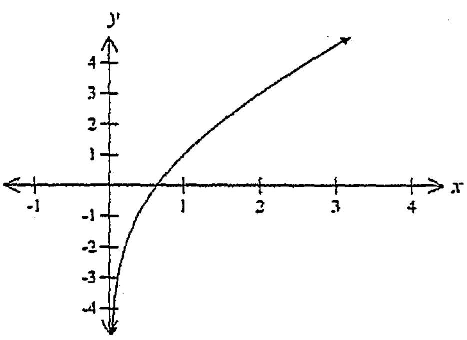
(A)



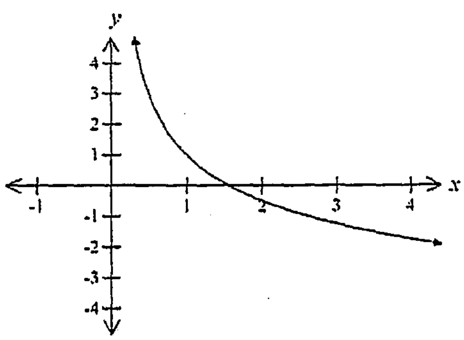
(B)



(C)



(D)



Student Name: ....

Question 11 (15 marks)

a) Let $w = \sqrt{3} + i$ and $z = 3 - \sqrt{3}\, i$

i) Find wz (1)

ii) Express w in mod/arg form (2)

iii) Write w^4 in simplest Cartesian form. (2)

b)

i) Mark clearly on an Argand diagram the region satisfied simultaneously by (2)

$|z + 2| < 2$ and $0 < \arg z < \frac{3\pi}{4}$

ii) Solve simultaneously (2)

$|z + 2| = 2$ and $\arg z = \frac{3\pi}{4}$

Write your answer in the form $a + ib$

c) A polynomial $P(x)$ has a double root at $x = \alpha$, ie $P(x) = (x - a)^2 Q(x)$

i) Prove that $P'(x)$ also has a root at $x = \alpha$ (2)

ii) The polynomial $Q(x) = x^4 - 6x^3 + ax^2 + bx + 36$ has a double root at $x = 3$ (2)

Find the values of a and b

iii) Factorise $Q(x)$ over the complex field. (2)

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Question 12 (15 marks)

a)

i) Show that $(\cos x - \sin x)^2 = 1 - \sin 2x$ (1)

ii) Evaluate (2)

$$\int_0^{\frac{\pi}{4}} \sqrt{1 - \sin 2x}\, dx$$

b) Using the substitution $u = 1 - x$, find (3)

$$\int x\sqrt{1 - x}\, dx$$

c)

i) Find the value of the integral (2)

$$\int_0^{\pi} \frac{1}{\sqrt{16 - x^2}}\, dx$$

ii) Find the integral of (2)

$$\int \frac{1}{16 - x^2}\, dx$$

d) If $I_n = \int_1^e x(lnx)^n dx$, $n = 0,1,2,3, \ldots\ldots$

i) Show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$, n=1, 2, 3, ... (3)

ii) Hence evaluate (2)

$$\int_1^e x(lnx)^3 dx$$

Student Name: ....

Question 13 (15 marks)

a)

i) Show that the equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $P(a\sec\theta, b\tan\theta)$ is $ax\sin\theta + by = (a^2 + b^2)\tan\theta$ (3)

ii) If the normal in part (i) intersects the x axis at A and the y axis at B, find the co-ordinates of A and B. (2)

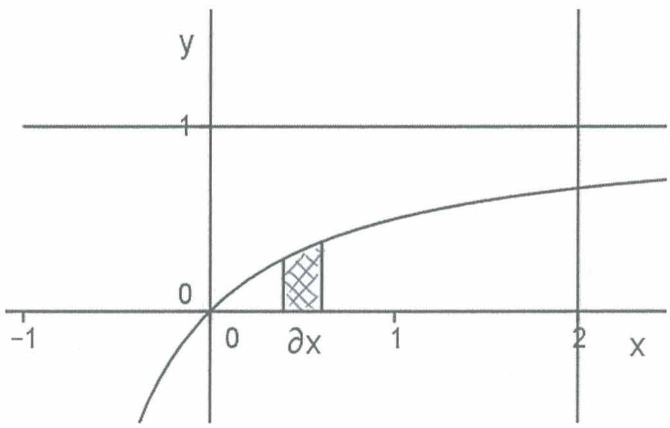
iii) Show that the co-ordinates of M, the midpoint of AB are given by (2)

$$x = \frac{1}{2a}(a^2 + b^2)\sec\theta, \qquad y = \frac{1}{2b}(a^2 + b^2)\tan\theta$$

iv) Hence find the equation of the locus of M in Cartesian form. (2)

v) If $a = b$, what can you say about the locus in part (iv) (1)

b) The region bounded by the portion of the curve $y = \frac{x}{x+1}$, and the x axis is rotated about the line $x = 2$



i) Using the method of cylindrical shells, show that the volume δV of a typical shell at a distance x from the origin and with thickness δx is given by (1)

$$\delta V = 2\pi(2 - x).\frac{x}{1 + x}.\delta x$$

ii) Hence find the volume of this solid. (4)

Student Name: ....

Question 14 (15 marks)

a) Consider the function $f(x) = (3 - x)(x + 1)$ on separate axes sketch, showing the important features the graphs of

i) $y = f(x)$ (1)

ii) $y = |f(x)|$ (1)

iii) $y = f|(x)|$ (1)

iv) $|y| = f(x)$ (1)

v) $y^2 = f(x)^3$ (2)

b) Given $a + b = m$, prove that, for $a > 0$, $b > 0$, and $m > 0$

i) $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{m}$ (2)

ii) $\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{m^2}$ (2)

c) Evaluate $\frac{lim}{\theta \to 0} \quad \frac{1 - \cos\theta}{\theta}$ (2)

d)

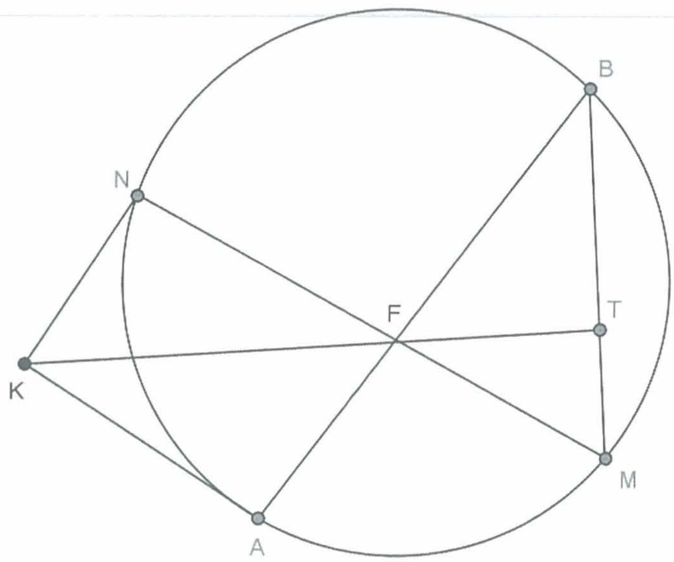
i) Use De Moivre's Theorem to express $\cos 3\theta$ and $\sin 3\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$ (2)

ii) Hence express $\tan 3\theta$ as a rational function of t, where $t = \tan\theta$ (1)

Student Name: ....

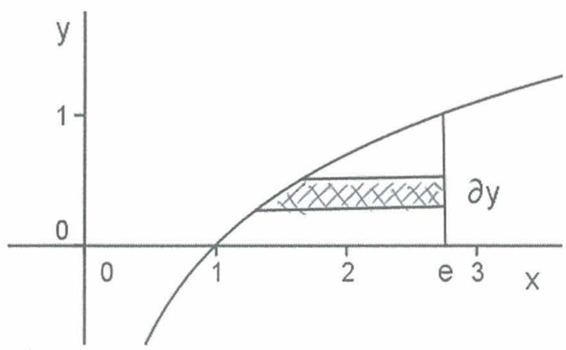
Question 15 (15 marks)

a) As shown below, a circle has two chords AB and MN intersecting at F. (4)
Perpendiculars are drawn to these chords at A and at N, intersecting at K.
KF produced, meets MB at T.
Prove that KT is perpendicular to MB (Hint: Join AN and let $\angle ANF = \theta^\circ$)



b) If $V_1 = 1, V_2 = 5$ and $V_n = 5V_{n-1} - 6V_{n-2}$ for $n \geq 3$, show that (3)
$V_n = 3^n - 2^n$ for $n \geq 1$

c) Consider the curve $y = \ln x$ sketched below.



Use the method of slicing to find the volume obtained by rotating the region bounded by (3)
$1 \leq x \leq e\ ,\ 0 \leq y \leq \ln x$, about the y axis.

Question 15 (Cont'd)

d) The equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β, γ. Find equations with roots

i) $2\alpha + \beta + \gamma,\ \alpha + 2\beta + \gamma,\ \alpha + \beta + 2\gamma$ (3)

ii) Find the value of the sum of the squares of the roots of the equation formed in (i) (2)

Student Name: ....

Question 16 (15 marks)

a) Find $\int sin^5\,\theta cos^4\theta\; d\theta$ (3)

b) If $x^2 + y^2 + xy = 3$

i) Find $\frac{dy}{dx}$ (2)

ii) Sketch showing critical points and stationary points the graph of $x^2 + y^2 + xy = 3$ (3)

c)

i) If $x_1 > 1$ and $x_2 > 1$ show that $x_1 + x_2 > \sqrt{x_1 x_2}$ (3)

ii) Use the Principal of Mathematical Induction to show that, (4)
For n ≥ 2 , if $x_j > 1$ where $j = 1, 2, 3, \dots \dots n$ then,

$$\ln(x_1 + x_2 + \cdots \dots \dots \dots . + x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \cdots \dots \dots . . + \ln x_n)$$

END OF PAPER

Find $\int x\sin(x^2+3)dx$

(A) $-\frac{1}{2}\cos(x^2+3)$+c

(B) $-\frac{1}{2}\sin(x^2+3)+c$

(C) $\frac{1}{2}\cos(x^2+3)+c$

(D) $2x\cos(x^2+3)+c$

If ω is a non-real cube root of unity the value of $\frac{1}{1+\omega}+\frac{1}{1+\omega^2}$ is equal to

(a) -1 (B) 0 (C) 1 (D) None of these

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Multiple Choice

1, C 2, C 3, D 4, B 5, B

6, B 7, A 8, C 9, C 10, A

Question 11

(a) (i) ωz

$= (\sqrt{3} - i)(3 - \sqrt{3}i)$

$= 3\sqrt{3} - 3i + 3i + \sqrt{3}$

$= 4\sqrt{3}$

(ii) $r = \sqrt{(\sqrt{3})^2 + (1)^2}$

$= 2$

$\tan\theta = \frac{1}{\sqrt{3}}$

$\theta = \pi/6$

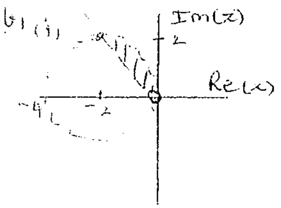
$\omega = 2\,\text{cis}\,\pi/6$

(iii) $\omega^4 = (2\,\text{cis}\,\pi/6)^4$

$= 2^4\,\text{cis}\,4\pi/6$

$= -8 + 8\sqrt{3}i$

(b) (i)



(ii) $A(-2 + 2i)$

(c) (i) $P(x) = (x-\alpha)^2 Q(x)$

$P'(x) = 2(x-\alpha)Q(x) + (x-\alpha)^2 . Q'(x)$

$= (x-\alpha)\left[2Q(x) + (x-\alpha)Q'(x)\right]$

$\therefore P'(x)$ has a factor $(x-\alpha)$

$\therefore x = \alpha$ is a root

(c) (ii) $P(x) = x^4 - 6x^3 + ax^2 + bx + 36$

$P(3) = 0$

$0 = 81 - 162 + 9a + 3b + 36$

$3a + b = 15$ — (1)

$P'(3) = 0$

$P'(x) = 4x^3 - 18x^2 + 2ax + b$

$0 = 4(27) - 18(9) + 6a + b$

$6a + b = 54$ — (2)

(1) − (2) $\quad -3a = -39$

$a = 13, \quad b = -24$

(c) (iii) $P(x) = x^4 - 6x^3 + 13x^2 - 24x + 36$

$P(x) = (x-3)^2(x^2 + mx + n)$

$= (x^2 - 6x + 9)(x^2 + mx + n)$

$= x^4 + x^3(m - 6) + x^2(-6m + n + 9) + \ldots$

$+ x(-6n + 9m) + 9n$

$\therefore 9n = 36 \qquad -6m + n + 9 = 13$

$n = 4 \qquad -6m + 13 = 13$

$m = 0$

Factors of $P(x)$

$P(x) = (x-3)^2(x^2+4)$

$= (x-3)^2(x+2i)(x-2i)$

OR

$$\begin{array}{r} x^2 + 4 \\ x^2 - 6x + 9 \,\big)\, \overline{x^4 - 6x^3 + 13x^2 - 24x + 36} \\ x^4 - 6x^3 + 9x^2 \\ \hline 4x^2 - 24x + 36 \\ 4x^2 - 24x + 36 \\ \hline 0 \end{array}$$

$\therefore P(x) = (x-3)^2(x^2+4)$

$= (x-3)^2(x-2i)(x+2i)$

Question 12

(a) (i) $(\cos x - \sin x)^2 = 1 - \sin 2x$

L.H.S.

$\cos^2 x - 2\sin x\cos x + \sin^2 x$

$= \cos^2 x + \sin^2 x - 2\sin x\cos x$

$= 1 - 2\sin x\cos x$

$= 1 - \sin 2x$

(ii) $\int_0^{\pi/4} \sqrt{1 - \sin 2x}\, dx$

$= \int_0^{\pi/4} \cos x - \sin x\, dx$

$= \left[\sin x + \cos x\right]_0^{\pi/4}$

$= \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (1)$

$= \sqrt{2} - 1$

(b) $\int x\sqrt{1-x}\, dx$

Let $u = 1 - x$

$x = 1 - u$

$\frac{du}{dx} = -1$

$dx = -du$

$= \int (1-u)u^{1/2} . -du$

$= \int (u-1)u^{1/2} . du$

$= \int u^{3/2} - u^{1/2}\, du$

$= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C$

(c) (i) $\int_0^{\pi} \frac{1}{\sqrt{16 - x^2}}\, dx$

$= \int_0^{\pi} \frac{1}{\sqrt{4^2 - x^2}}\, dx$

$= \left[\sin^{-1}\frac{x}{4}\right]_0^{\pi}$

$\doteqdot 0.903$

(c) (ii) $\int \frac{1}{16 - x^2}\, dx$

$= \frac{1}{8}\int \frac{1}{4-x} + \frac{1}{4+x}\, dx$

$= \frac{1}{8}\left[-\ln|4-x| + \ln|4+x|\right]$

$= \frac{1}{8}\ln\left|\frac{4+x}{4-x}\right| + C$

$\frac{1}{(4-x)(4+x)} = \frac{A}{4-x} + \frac{B}{4+x}$

$= \frac{4A + xA + 4B - xB}{(4-x)(4+x)}$

$4A + 4B = 1 \qquad A - B = 0$

$A = B$

$A = 1/8, \quad B = 1/8$

(d) $I_n = \int_1^e x(\ln x)^n dx \quad n = 0, 1, 2$

Let $u = (\ln x)^n, \quad v' = x$

$v = x^2/2$

$I_n = uv - \int v\, du$

$= \left[(\ln x)^n . \frac{x^2}{2}\right]_1^e - \int_1^e \frac{x^2}{2} n(\ln x)^{n-1} . \frac{1}{x}$

$= \left[(\ln x)^n . \frac{x^2}{2}\right]_1^e - \frac{n}{2}\int_1^e x(\ln x)^{n-1}\, dx$

Question 12 (cont'd)

(b) $I_n = \left[(\ln e)^n \cdot \frac{e^2}{2}\right] - \frac{n}{2}\int I_{n-1}$

$I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$

(ii) $\int_1^e x(\ln x)^3\,dx$

$I_n = \frac{e^2}{2} - \frac{3}{2}\left[\int_1^e x(\ln x)^2\,dx\right]$

$= \frac{e^2}{2} - \frac{3}{2}\left[\frac{e^2}{2} - I_1\right]$

$= \frac{e^2}{2} - \frac{3}{2}\cdot\frac{e^2}{2} + \frac{3}{2} I_1$

$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3}{2}\left[\frac{e^2}{2} - \frac{1}{2}e^0\right]$

$= \frac{e^2}{2} - \frac{3e^2}{4} + \frac{3e^2}{4} - \frac{3}{4}\int_1^e x\,dx$

$= \frac{e^2}{2} - \frac{3}{4}\left[\frac{x^2}{2}\right]_1^e$

$= \frac{e^2}{2} - \frac{3}{4}\left[e^2/2 - 1/2\right]$

$= \frac{e^2}{2} - \frac{3e^2}{8} + \frac{3}{8}$

$= \frac{e^2+3}{8}$

Question 13

(a)(i) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{dy}{dx} = \frac{b^2x}{a^2y}$

Grad of normal

$= -\frac{a^2y}{b^2x}$

at P $= \frac{-a^2b\tan\theta}{b^2a\sec\theta}$

$= -\frac{a}{b}\sin\theta$

∴ Eqn of normal

$y - b\tan\theta = -\frac{a}{b}\sin\theta(x - a\sec\theta)$

$by - b^2\tan\theta = -ax\sin\theta + a^2\sin\theta\sec\theta$

$ax\sin\theta + by = (a^2+b^2)\tan\theta$

(ii) At A, y=0

$x = \frac{a^2+b^2}{a\sin\theta}\cdot\tan\theta$

$= \frac{a^2+b^2}{a}\sec\theta$

At B, x=0

$y = \frac{a^2+b^2}{b}\tan\theta$

(iii) $x = \frac{1}{2}\left[\frac{a^2+b^2}{a}\sec\theta + 0\right]$

$= \frac{a^2+b^2}{2a}\sec\theta$ —(1)

$y = \frac{1}{2}\left[\frac{a^2+b^2}{b}\tan\theta + 0\right]$

$y = \frac{a^2+b^2}{2b}\tan\theta$ —(2)

Question 13 (cont'd)

from (iii) (1) and (2)

(iv) $\frac{2ax}{a^2+b^2} = \sec\theta$ — θ

$\frac{2by}{a^2+b^2} = \tan\theta$ — ϕ

$\theta^2 - \phi^2$

$= \frac{4a^2x^2}{(a^2+b^2)^2} - \frac{4b^2y^2}{(a^2+b^2)^2} = 1$

$4a^2x^2 - 4b^2y^2 = (a^2+b^2)^2$

(v) If $a=b$

$4a^2x - 4a^2y^2 = 4a^4$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

when $a=b$

$x^2 - y^2 = a^2$

Rectangular Hyperbola

(b) $\delta V = \left[\pi(2-x)^2 - \pi\left[2-(x+\delta x)\right]^2\right]\cdot\frac{x}{x+1}$

$\delta V = \pi\left\{(2-x)^2 - \left[2-(x+\delta x)\right]^2\right\}\cdot\frac{x}{x+1}$

$= \pi\left\{\delta x\left[4-2x-\delta x\right]\right\}\cdot\frac{x}{x+1}$

$= 2\pi\left\{(2-x)\,\delta x\right\}\cdot\frac{x}{x+1}$

(b) $V = 2\pi\int_0^2 (2-x)\frac{x}{x+1}\,\delta x$

$V = 2\pi\int_0^2 -x+3-\frac{3}{x+1}\,\delta x$

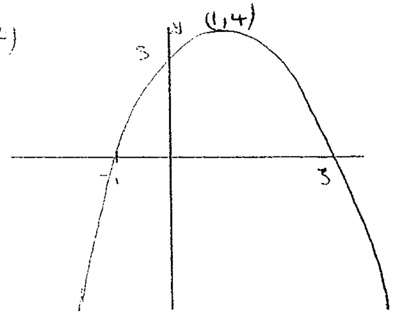
$V = 2\pi\left[-\tfrac{1}{2}x^2+3x-3\ln(x+1)\right]_1^2$

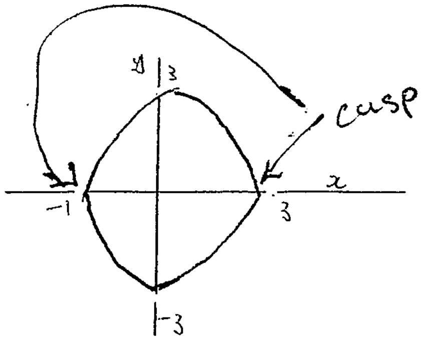
$V = 2\pi\left[(-2+6-3\ln 3)-0\right]$

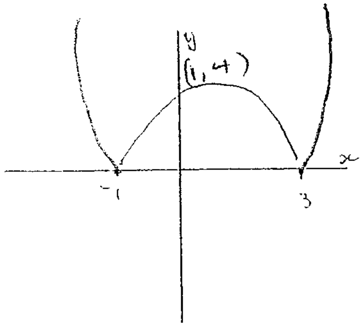
$V = 2\pi\left[4-3\ln 3\right]\ u^3$

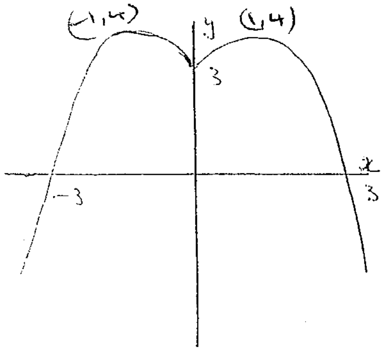
Question 14

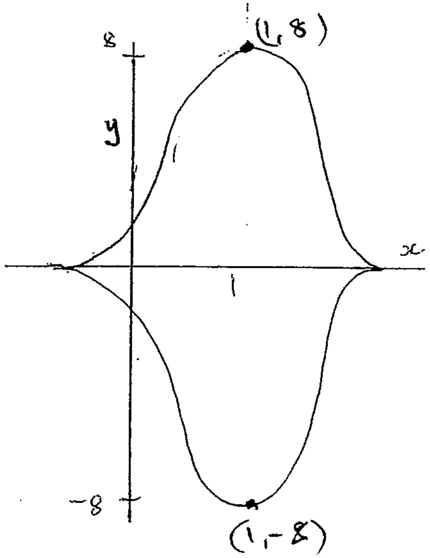
(a)











Question 14

(b) (i) $\frac{1}{a} + \frac{1}{b} - \frac{4}{m} = \frac{1}{a} + \frac{1}{b} - \frac{4}{a+b}$

R.H.S. $\frac{1}{a} + \frac{1}{b} - \frac{4}{a+b}$

$= \frac{b(a+b) + a(a+b) - 4ab}{ab(a+b)}$

$= \frac{(a+b)^2 - 4ab}{ab(a+b)}$

$= \frac{a^2 - 2ab + b^2}{ab(a+b)}$

$= \frac{(a-b)^2}{ab(a+b)}$

now since $a>0,\ b>0,\ (a-b)^2>0$

$ab(a+b) > 0$ for all a, b

Hence $\frac{1}{a} + \frac{1}{b} - \frac{4}{m} > 0$

$\frac{1}{a} + \frac{1}{b} \geq \frac{4}{m}$

(ii) $x^2 + y^2 \geq 2xy$

Let $x = \frac{1}{a}$, $y = \frac{1}{b}$

$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{ab}$ — (1)

from (i) $\frac{1}{a} + \frac{1}{b} \geq \frac{4}{m}$

$\frac{a+b}{ab} \geq \frac{4}{m}$ $(a+b=m)$

$\frac{1}{ab} \geq \frac{4}{m^2}$

(ii) from (1)

$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{2}{\frac{m^2}{4}}$

$\frac{1}{a^2} + \frac{1}{b^2} \geq \frac{8}{m^2}$.

(c) $\lim_{\theta \to 0} \frac{1 - \cos\theta}{\theta}$

$= \frac{1-\cos\theta}{\theta} \times \frac{1+\cos\theta}{1+\cos\theta}$

$= \frac{1-\cos^2\theta}{\theta(1+\cos\theta)}$

$= \frac{\sin^2\theta}{\theta(1+\cos\theta)}$

$= \frac{\sin\theta}{\theta} \times \frac{\sin\theta}{1+\cos\theta}$

$= 1 \times 0$

$= 0$

Question 14 cont'd

(i) $(\text{cis}\,\theta)^3 = (\cos\theta + i\sin\theta)^3$

$(\text{cis}\,\theta)^3 = \cos^3\theta - 3\sin^2\theta\cos\theta + 3i\cos^2\theta\sin\theta - i\sin^3\theta$

$= \cos^3\theta - 3\sin^2\theta\cos\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)$

Equate real & Imaginary

$\cos 3\theta = \cos^3\theta - 3\sin^2\theta\cos\theta$ — ①

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ — ②

$\tan 3\theta = \frac{②}{①}$

$\tan 3\theta = \frac{3\cos^2\theta\sin\theta - \sin^3\theta}{\cos^3\theta - 3\sin^2\theta\cos\theta}$

÷ RHS top & bottom by $\cos^3\theta$

$= \frac{\frac{3\sin\theta}{\cos\theta} - \frac{\sin^3\theta}{\cos^3\theta}}{1 - 3\frac{\sin^2\theta}{\cos^2\theta}}$

$= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

$= \frac{3t - t^3}{1 - 3t^2}$

where $t = \tan\theta$

Question 15

(a) $\angle ANM = \angle ABM = \theta$ (angles at circumference on same chord AM)

$\angle KNF + \angle KAF = 180°$

∴ KNFA is a cyclic quad

$\angle AKF = \angle ANF = \theta$ (angles at circumference on same chord of circle through KNFA)

$\angle KFA = 90 - \theta$ (angle sum ΔAKF)

$\angle BFT = \angle KFA = 90 - \theta$ (vert opp $\angle$'s)

$\angle FTB = 180 - (\angle BFT + \angle FBM)$

$\angle ABM = \angle FBM$ (same angle)

$\angle FTB = 180 - [(90 - \theta) + \theta]$

$= 90°$

∴ $KF \perp MB$

(b) $V_1 = 1$, $V_2 = 5$ $\quad V_n = 3^n - 2^n$, $n \geq 1$

$V_n = 5V_{n-1} - 6V_{n-2}$

$V_1 = 3^1 - 2^1$
$= 1$

$V_2 = 3^2 - 2^2$
$= 5$

Assume true for $n = k$

$V_k = 3^k - 2^k \quad k \geq 1$

Question 15 (cont'd)

Hence true for $n = k+1$

$V_{k+1} = 5V_{k+1-1} - 6V_{k+1-2}$

$= 5V_k - 6V_{k-1}$

$= 5[3^k - 2^k] - 6[3^{k-1} - 2^{k-1}]$

$= 5.3^k - 5.2^k - 6\left[\frac{2.3^k - 3.2^k}{6}\right]$

$= 5.3^k - 5.2^k - 2.3^k + 3.2^k$

$= 3.3^k - 2.2^k$

$= 3^{k+1} - 2^{k+1}$

Plus statement on M.I. proof

$\delta V = [\pi e^2 - \pi(e^y)^2]\,\delta y$

$\left[\begin{array}{l} y = \log_e x \\ e^y = x \end{array}\right]$

$\delta V = \pi[e^2 - e^{2y}]\,\delta y$

$V = \pi \int_0^1 e^2 - e^{2y}\, dy$

$V = \pi\left[ye^2 - \frac{1}{2}e^{2y}\right]_0^1$

$V = \pi\left[(e^2 - \frac{1}{2}e^2) - (0 - \frac{1}{2})\right]$

$= \pi\left[\frac{1}{2}e^2 + \frac{1}{2}\right]$

$= \frac{\pi}{2}[e^2 + 1]\ u^3$

Question 15

(d) $x^3 - 3x^2 - x + 2 = 0$ — (1)

$\alpha + \beta + \gamma = 3$.

(i) $2\alpha + \beta + \gamma,\ \alpha + 2\beta + \gamma,\ \alpha + \beta + 2\gamma$

$x = \alpha + \alpha + \beta + \gamma$

$x = \alpha + 3$

$\therefore \alpha = x - 3$

sub into (1)

$(x-3)^3 - 3(x-3)^2 - (x-3) + 2 = 0$

$x^3 - 12x^2 + 44x - 49 = 0$

(ii) $\alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= \left(\frac{12}{1}\right)^2 - 2\left(\frac{44}{1}\right)$

$= 144 - 88$

$= 56$

Question 16

$\int \sin^5\theta \cos^4\theta \, d\theta$

$= -\int \sin^4\theta \cos^4\theta (-\sin\theta) \, d\theta$

$\sin^4\theta$
$= (1-\cos^2\theta)^2$
$= 1 - 2\cos^2\theta + \cos^4\theta$

$= -\int \cos^4\theta(-\sin\theta)\,d\theta + 2\int \cos^6\theta(-\sin\theta)\,d\theta - \int \cos^8\theta(-\sin\theta)\,d\theta$

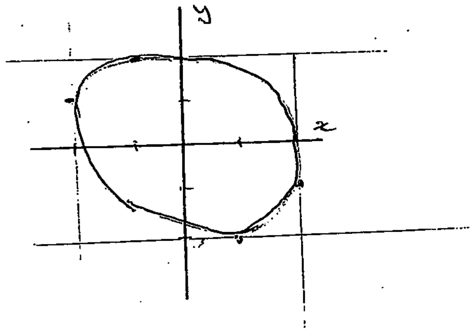
$= -\frac{1}{5}\cos^5\theta + \frac{2}{7}\cos^7\theta - \frac{1}{9}\cos^9\theta + C$

$x^2+y^2+xy=3$

$\frac{dy}{dx} = -\frac{(2x+y)}{x+2y}$

The tangents at the critical points $(-1,2)$ and $(1,-2)$ are horizontal

The tangents at the critical points $(-2,1)$ and $(2,-1)$ are vertical



Question 16

(c) (i) $x_1 > 1$, $x_2 > 1$

Consider

$(\sqrt{x_1} - \sqrt{x_2})^2 > 0$

$x_1 - 2\sqrt{x_1x_2} + x_2 > 0$

$x_1 + x_2 > 2\sqrt{x_1x_2}$

$x_1 + x_2 > \sqrt{x_1x_2}$

(c) (ii)

To prove $\ln(x_1+x_2+\dots+x_n) > \frac{1}{2^{n-1}}(\ln x_1 + \ln x_2 + \dots + \ln x_n)$

when $n=2$ we know

$x_1+x_2 > \sqrt{x_1x_2}$

$\ln(x_1+x_2) > \ln\sqrt{x_1x_2}$ since x_1, x_2 are both > 1

$\ln(x_1+x_2) > \ln(x_1x_2)^{1/2}$

$\ln(x_1+x_2) > \frac{1}{2}\ln(x_1x_2)$

$\ln(x_1+x_2) > \frac{1}{2}(\ln x_1 + \ln x_2)$

Assume true for $n=k$

$\ln(x_1+x_2+\dots x_k) > \frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \dots + \ln x_k)$

when $n=k+1$

L.H.S. $\ln(x_1+x_2+\dots x_k + x_{k+1}) > \frac{1}{2}[\ln(x_1+x_2+\dots+x_k) + \ln x_{k+1}]$

$> \frac{1}{2}\left[\frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \dots \ln x_k) + \ln x_{k+1}\right]$

NB $\left[\frac{1}{2^{k-1}} < 1\right]$

$> \frac{1}{2}\left[\frac{1}{2^{k-1}}(\ln x_1 + \ln x_2 + \dots \ln x_k + \frac{1}{2^{k-1}}\ln x_{k+1}\right]$

$\ln(x_1+x_2+\dots+x_k+x_{k+1}) > \frac{1}{2^k}\left[\ln x_1 + \ln x_2 + \dots \ln x_k + \ln x_{k+1}\right]$

* Plus M.I statement

Multiple Choice. Solutions

1/ $\frac{1}{1+\omega} + \frac{1}{1+\omega^2}$ $\qquad 1+\omega+\omega^2=0$

$\omega = 1$

$= \frac{1+\omega^2 + 1+\omega}{(1+\omega)(1+\omega^2)}$

$= \frac{1}{1+\omega^2+\omega+\omega^3}$

$= \frac{1}{1}$

$= 1$ (C)

2/ $P(x) = x^3+x^2+5x+6$

$P(-i) = (-i)^3+(-i)^2+5(-i)+6$

$= i-1-5i+6$

$= 5-4i$ (C)

3/ $xy^3+2y=4$

$y^3.1.dx + x.3y^2.dy + 2.dy = 0$

$y^3 + 3xy^2\frac{dy}{dx} + 2\frac{dy}{dx} = 0$

$y^3 + \frac{dy}{dx}(3xy^2+2) = 0$ (D)

$\frac{dy}{dx} = \frac{-y^3}{3xy^2+2}$

when $x=2$, $y=1$ $\quad \frac{dy}{dx} = \frac{-1}{8}$

4) $3x^2 + 5y^2 - 15 = 0$

$\frac{x^2}{5} + \frac{y^2}{3} = 1$

$b^2 = a^2(1 - e^2)$

$\frac{3}{5} = 1 - e^2$

$e = \sqrt{2/5}$ (B)

5) Let $y = \frac{2}{\alpha}$

$\alpha = \frac{2}{y}$

$3\left(\frac{2}{y}\right)^3 - 2\left(\frac{2}{y}\right)^2 + \frac{2}{y} - 7 = 0$

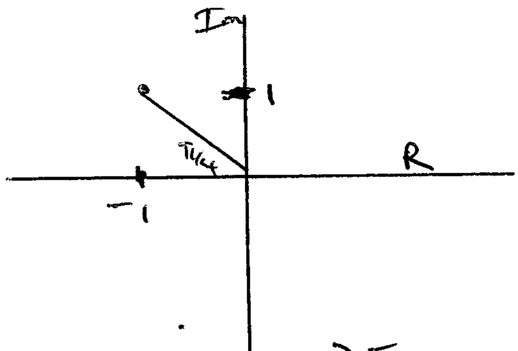
$\frac{24}{y^3} - \frac{8}{y^2} + \frac{2}{y} - 7 = 0$

$24 - 8y + 2y^2 - 7y^3 = 0$

$7y^3 - 2y^2 + 8y - 24 = 0$ (B)

6) $z = 1 + i$

$iz = i - 1$



$\arg iz = \frac{3\pi}{4}$ (B)

7. $\int x \sin(x^2+3)\,dx$

$= -\frac{1}{2}\cos(x^2+3) + C$ (A)

8. Roots in conjugate pairs since coefficients real. ∴ 3

(C)

9. $\int_0^1 \frac{e^x}{1+e^x}$

$= \left[\ln(1+e^x)\right]_0^1$

$= \ln(1+e) - \ln 2$

$= \ln\left(\frac{1+e}{2}\right)$ (C)

10. (A)

EXT 2 trial mark breakdown 2014   NAME

COMPLEX NUMBERS

question	mark
1	/1
2	/1
6	/1
11ai	/1
11aii	/2
11aiii	/2
11bi	/2
11bii	/2
14di	/2
14dii	/1
TOTAL	/15

CONICS

question	mark
3	/1
4	/1
13ai	/3
13aii	/2
13aiii	/2
13aiv	/2
13av	/1
TOTAL	/12

GRAPHS

question	mark
10	/1
14ai	/1
14aii	/1
14aiii	/1
14aiv	/1
14av	/2
16bi	/2
16bii	/3
TOTAL	/12

POLYNOMIALS

question	mark
5	/1
8	/1
11ci	/2
11cii	/2
11ciii	/2
15di	/3
15dii	/2
TOTAL	/13

VOLUMES

question	mark
13bi	/1
13bii	/4
15c	/3
TOTAL	/8

INTEGRATION

question	mark
7	/1
9	/1
12ai	/1
12aii	/2
12b	/3
12ci	/2
12cii	/2
12di	/3
12dii	/2
16a	/3
TOTAL	/20

HARDER 3U

question	mark
14bi	/2
14bii	/2
14c	/2
15a	/4
15b	/3
16ci	/3
16cii	/4
TOTAL	/20

SUMMARY

COMPLEX NUMBERS	/15
CONICS	/12
GRAPHS	/12
POLYNOMIALS	/13
VOLUMES	/8
HARDER 3U	/20
INTEGRATION	/20
TOTAL	/100